Lecture XIX: Bose-Einstein Condensation

Previously, we have seen how the functional field integral technique can be developed to explore the impact of the electronic degrees of freedom on the effective Coulomb interaction in a metal. However, our considerations did not engage any non-trivial mean-fields: the platform of the non-interacting electron system remains adiabatically connected to that of the weakly interacting system. In the following we will explore a problem in which the development of a non-trivial ground state — the Bose-Einstein condensate — is accompanied by the appearance of collective modes absent in the non-interacting system.

▷ Following our earlier considerations, we begin with a Hamiltonian describing a Bose system of size L subject to a weak short-ranged repulsive contact interaction:

$$\hat{H} = \int d^d r\, a^\dagger(\mathbf{r})\hat{H}_0 a(\mathbf{r}) + \frac{g}{2}\int d^d r\, a^\dagger(\mathbf{r})a^\dagger(\mathbf{r})a(\mathbf{r})a(\mathbf{r})$$

▷ CSPI: $\mathcal{Z} = \mathrm{tr} e^{-\beta(\hat{H}-\mu\hat{N})} = \int_{\psi(\beta)=\psi(0)} D(\bar\psi,\psi)\, e^{-S[\bar\psi,\psi]}$,

$$S = \int_0^\beta d\tau \int d^d r\left[\bar\psi(\mathbf{r},\tau)(\partial_\tau + \hat{H}_0 - \mu)\psi(\mathbf{r},\tau) + \frac{g}{2}(\bar\psi(\mathbf{r},\tau)\psi(\mathbf{r},\tau))^2\right],$$

▷ BOSE-EINSTEIN CONDENSATION

As a warm-up, consider non-interacting Bose gas with spectrum ϵ_a

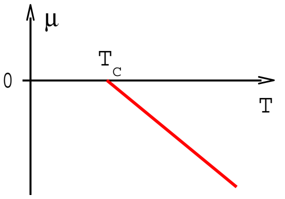
$$\mathcal{Z}_0 \equiv \mathcal{Z}\Big|_{g=0} = \int_{\text{p.b.c.}} D(\bar\psi,\psi)\, e^{-\sum_{a\omega_n}\bar\psi_{a,\omega_n}(-i\omega_n+\epsilon_a-\mu)\psi_{a\omega_n}} = \text{const.}\prod_{a,\omega_n}\frac{1}{-i\omega_n+\epsilon_a-\mu}$$

where, w.l.o.g., we assume $\epsilon_a \geq 0$ and $\epsilon_0 = 0$

While stability of integral requires $\mu \leq 0$, precise value fixed by

$$N(\mu) = -\partial_\mu F = \frac{1}{\beta}\partial_\mu \ln \mathcal{Z}_0 = -\frac{1}{\beta}\sum_{a,\omega_n}\frac{1}{i\omega_n - \epsilon_a + \mu} = \sum_a n_{\mathrm{B}}(\epsilon_a),$$

where, using Matsubara summation, $n_{\mathrm{B}}(\epsilon) = \dfrac{1}{e^{\beta(\epsilon-\mu)}-1}$ (Bose-Einstein distribution)



As T reduced, μ increased until, at $T = T_{\mathrm{BEC}}$, $\mu = 0$. For $T < T_{\mathrm{BEC}}$, μ remains zero and a macroscopic number of particles $N - N_1$ condense into ground state: Bose-Einstein condensation

$$\text{i.e. for } T < T_{\mathrm{BEC}}, \qquad \sum_{a>0} n_{\mathrm{B}}(\epsilon_a)\Big|_{\mu=0} \equiv N_1 < N$$

▷ How can this phenomenon be incorporated into the path integral?

Although condensate characterised by ground state component $\psi_0(\tau)$ for $T < T_{\rm BEC}$, fluctuations are unbound ($\mu = 0 = \epsilon_0$ and action for $\psi_{0,0}$ vanishes!)

Here we must treat field $\psi_0(\tau)$ as a (time-independent) Lagrange multiplier to be used to fix the number of particles below $T_{\rm BEC}$:

$$S_0|_{\mu=0^-} \simeq -\beta\bar{\psi}_0\mu\psi_0 + \sum_{a\neq 0,\omega_n} \bar{\psi}_{a\omega_n}\left(-i\omega_n + \epsilon_a - \mu\right)\psi_{a\omega_n}$$

$$\text{i.e. } N = -\partial_\mu F|_{\mu=0^-} = \frac{1}{\beta}\partial_\mu \ln \mathcal{Z}_0|_{\mu=0^-} = \bar{\psi}_0\psi_0 - \frac{1}{\beta}\sum_{a\neq 0,\omega_n}\frac{1}{i\omega_n - \epsilon_a} = \bar{\psi}_0\psi_0 + N_1$$

i.e. $\bar{\psi}_0\psi_0 = N_0$ translates to number of particles in condensate

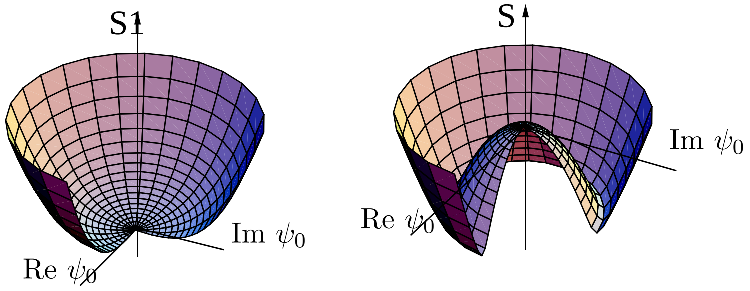
▷ WEAKLY INTERACTING BOSE GAS

As in electron gas Presence of interaction prevents exact evaluation of field integral therefore, turn to saddle-point + fluctuations analysis

▷ LANDAU THEORY

In saddle-point (mean-field) approximation dominant contribution to $\mathcal{Z} = \int D(\bar{\psi},\psi)e^{-S}$ controlled by $\psi_0(\omega_n = 0)$ (i.e. time-independent) sector of theory

$$\boxed{\frac{1}{\beta}S[\bar{\psi}_0,\psi_0] = -\mu\bar{\psi}_0\psi_0 + \frac{g}{2L^d}(\bar{\psi}_0\psi_0)^2}$$



Minimum action obtained from saddle-point equation: $\psi_0(-\mu + \frac{g}{L^d}\bar{\psi}_0\psi_0) = 0$

- ▷ For $\mu < 0$ (i.e. above the condensation threshold of the non-interacting system), s.p.e. exhibits only trivial solution $\psi_0 = 0$ — no stable condensate
- ▷ Below condensation threshold (i.e. for $\mu \geq 0$), equation solved by any configuration with $|\psi_0| = \gamma \equiv \sqrt{\mu L^d/g}$

N.B. $\bar{\psi}_0\psi_0 \propto L^d$, reflecting the macroscopic population of the ground state

- ▷ The equation couples only to the modulus of ψ_0, i.e. the solution of the stationary phase equation is "continuously degenerate": Each configuration $\psi_0 = \gamma \exp(i\phi), \phi \in [0, 2\pi]$ is a solution. One ground state chosen $\rightsquigarrow$ spontaneous symmetry breaking.

Self-consistent calculation of $\mu = \mu(N)$ demands consideration
of low-energy fluctuations around the mean-field solution

By taking into account such fluctuations, we will be also able
to address the phenomenon of superfluidity